Elementary proof of Pavillet tetrahedron properties

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Abstract

In this paper, we describe methods of elementary proof for the properties of the Pavillet tetrahedron. This solid gives students a great opportunity for in-depth study of perpendicularity in space and testing of various methods of proof. We have derived new methods for proving some theorems of plane triangle geometry, for example, the existence of the Gergonne point, the perpendicularity of Soddy line and Gergonne line.

1 Introduction

The orthocentric tetrahedron of triangle [8], [7] is a simple construction which gives students a great opportunity for in-depth study of perpendicularity in space and testing of various methods of proof. In this paper, we give elementary proofs of the main properties of a Pavillet tetrahedron. We use simple and sometimes coarse methods of proofs. Students may create their own short and elegant methods based on inversion or other transformations of the space.

2 Description of a Pavillet tetrahedron and its main property

Let's consider a triangle ABC, named the base triangle, be given. Let its incircle be centered in I and $A_1B_1C_1$ be the contact triangle of ABC. Let segments AA', BB', CC' be perpendicular to the plane ABC, where $AA' = AB_1, BB' = BC_1, CC' = CA_1$. Let us name the solid $IA'B'C'$ as the Pavillet tetrahedron of the triangle ABC [9, § 2], [3]. Let us name triangle $A'B'C'$ as upper triangle. This tetrahedron is shown on figure 1.

3 The tetrahedron is orthocentric

Theorem 1 *The segment $B'C'$ is perpendicular to the plane $IA'A_1$.*

Proof.

Let A_1' be the foot of the perpendicular dropped from A_1 on $B'C'$ (fig. 2) and denote $x = AB_1 = AC_1 = AA'$, $y = BC_1 = BA_1 = BB'$ and $z = CA_1 = CB_1 = CC'$. Hence

$$A_1B'^2 = 2\,y^2, A_1C'^2 = 2\,z^2 \Rightarrow A'B'^2 = (x+y)^2 + (x-y)^2 = 2\,(x^2+y^2).$$

We use $A_1'A_1 \perp B'C'$ and we get

$$A_1'B'^2 - A_1'C'^2 = A_1B'^2 - A_1C'^2 = 2\,(y^2 - z^2)$$

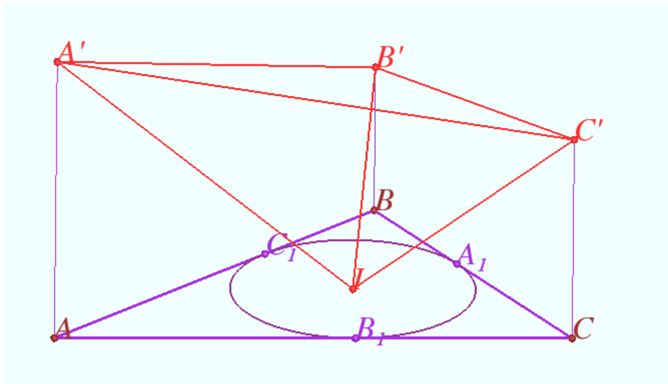


Figure 1: The Pavillet Tetrahedron of ABC

(Use mouse click to activate an interactive figure).

We calculate $A'B'^2 - A'C'^2 = 2\,(y^2 - z^2)$, it implies similarly that $A'A_1'$ is an altitude of the triangle $A'B'C'$. Now we also have $B'I^2 - C'I^2 = 2\,(y^2 - z^2)$. Hence $IA_1' \perp B'C'$, combined with $A_1'A_1 \perp B'C'$ it yields that the points I, A', A_1, lie on the same plane perpendicular to $B'C'$. This plane contains the point A_1' on $B'C'$.

We proved that the plane $IA'A_1'$ is perpendicular to the boundary plane $BB'C'$. ■

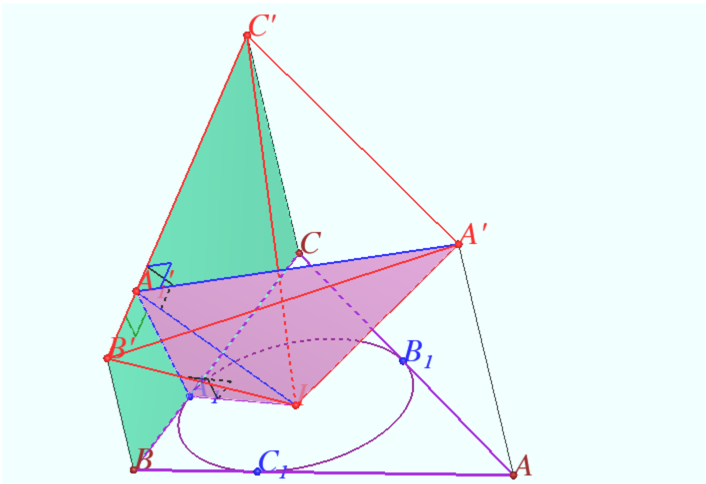


Figure 2: Plane $IA'A_1'$ is perpendicular to boundary plane $BCB'C'$

(Use mouse click to activate an interactive figure).

Theorem 2 *The Pavillet tetrahedron is orthocentric.*

Proof.

From Theorem 1, the plane $IA'A_1'$ is perpendicular to the boundary plane $BB'C'$. Therefore $IA' \perp B'C'$, similarly we get $IB' \perp C'A'$ and $IC' \perp A'B'$. From [1, Definition 208, p. 62] this tetrahedron is orthocentric. ■

4 Altitudes of the tetrahedron

Theorem 3 *The straight line joining a vertex of the upper triangle to the corresponding vertex of the contact triangle of the base triangle (e.g. $A'A_1$) is an altitude of the tetrahedron.*

Proof.

- It is clear, that $A'A_1 \perp B'C'$ as a line belonging to the plane $IA'A'_1$.
- We consider the quadrilateral $IA'A'_1A_1$ and compute the quantity $A'I^2 - A_1I^2$. We get

$$A'I^2 - A_1I^2 = A'B_1^2 + B_1I^2 - A_1I^2 = A'B_1^2 = 2\,x^2. \tag{1}$$

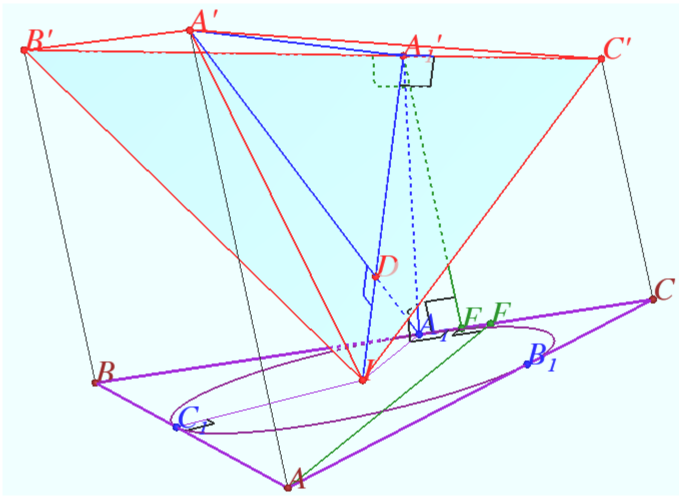


Figure 3: The line $A'A_1$ is perpendicular to the plane $IB'C'$

(Use mouse click to activate an interactive figure).

- In the same quadrilateral, we compute the quantity $A'A_1'^2 - A_1A_1'^2$. Let E and F lie on BC such that $A'_1E \perp BC$, $AF \perp BC$ (fig. 3). Then

$$A'A_1'^2 - A_1A_1'^2 = (A'A - A'_1E)^2 + AF^2 + EF^2 - A_1A_1'^2.$$

We use the right triangle $A_1B'C'$ with legs $A_1B' = y\sqrt{2}$, $A_1C' = z\sqrt{2}$. We get the square of the height

$$A_1A_1'^2 = \frac{2\,y^2\,z^2}{y^2+z^2},\, B'A_1'^2 = \frac{2\,y^4}{y^2+z^2},\, C'A_1'^2 = \frac{2\,z^4}{y^2+z^2}.$$

We use the trapezium (*British definition*) $BB'C'C$ and get

$$A'_1E = \frac{y\,z\,(y+z)}{y^2+z^2}.$$

From the right triangle $A_1A'_1E$, we get the leg $A_1E = \frac{y\,z\,|y-z|}{y^2+z^2}$. We use the formulas for the area of triangle ABC and get the square of the altitude $AF^2 = \frac{4\,x\,y\,z\,(x+y+z)}{(y+z)^2}$. We use the formula for distance between base of height and tangent point for incircle and get $A_1F = \frac{x\,|y-z|}{y+z}$. Finally, we use formula for square of incircle radius $A_1I^2 = C_1I^2 = \frac{x\,y\,z}{x+y+z}$.

After substitutions and simplifications we get

$$A'A_1'^2 - A_1A_1'^2 = 2x^2.$$

- We recall, to finish the proof, that *"for any orthodiagonal quadrilateral, the sum of the squares of two opposite sides equals that of the other two opposite sides and conversely"*.

  Now, because we have found that $2\,x^2 = A'I^2 - A_1I^2 = A'A_1'^2 - A_1A_1'^2$, the sides of the quadrilateral $IA'A_1'A_1$ comply with the equation

  $$A'I^2 + A_1A_1'^2 = A'A_1'^2 + A_1I^2$$

  so that $IA'A_1'A_1$ is orthodiagonal and $A'A_1 \perp IA_1'$.

We have proved that $A'A_1$ being perpendicular to IA_1' and $B'C'$, hence to the plane $IB'C'$, is an altitude of $IA'B'C'$. ■

4.1 Orthogonal Projection of the Orthocenter

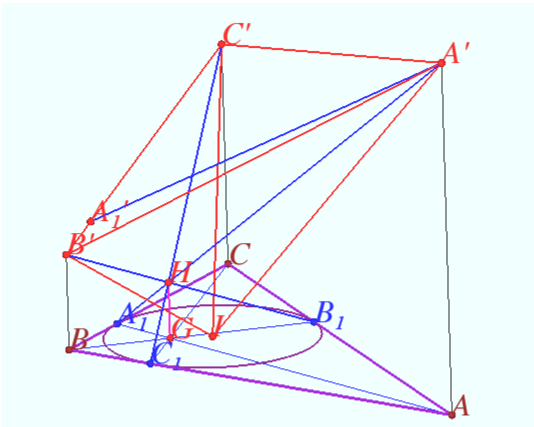


Figure 4: Orthogonal Projection of the Orthocenter

(Use mouse click to activate an interactive figure).

Theorem 4 *The orthogonal projection of the orthocenter of the Pavillet tetrahedron on the plane of its base triangle is the Gergonne point of this base triangle.*

Proof.

Theorem 3 shows that the lines $A'A_1$, $B'B_1$ and $C'C_1$ are three altitudes of the orthocentric tetrahedron. They intersect at the point H_o orthocenter of the tetrahedron (fig. 4). Each of these altitudes lies in a plane perpendicular to the plane $A'B'C'$, e.g. A_1A' in the plane AA_1A' which contain $AA' \perp ABC$. Let G be the point of intersection of the lines AA_1 and BB_1, orthogonal projections of the lines $A'A_1$ and $B'B_1$ on the base plane ABC. Then H_0G is perpendicular to the base plane.

Now, because $C'C_1$ also goes through G, the line CC_1 which is the orthogonal projection of $C'C_1$ is concurrent with AA_1 and BB_1.

As the reader sees, we have proved the existence of the Gergonne point. We also have proved that the orthogonal projection of the orthocenter of the Pavillet tetrahedron on the plane of its base triangle is the Gergonne point of its base triangle. ■

5 Axis of perspective

Theorem 5 *The corresponding sides of the base triangle, the upper triangle and the contact triangle are concurrent.*

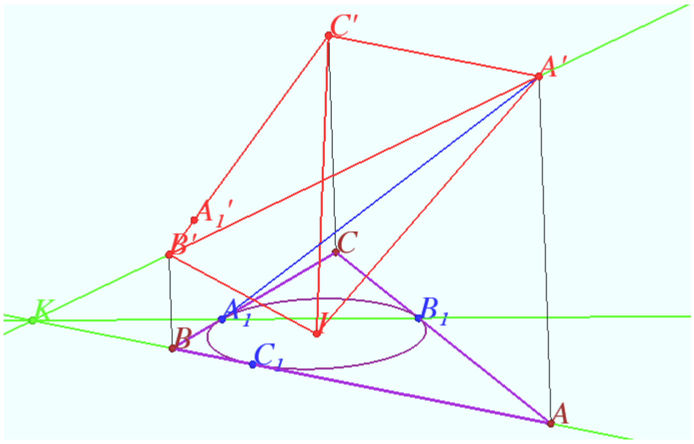


Figure 5: Three concurrent lines

(Use mouse click to activate an interactive figure).

Proof.

We use Menelaus' theorem for the sides of the triangle ABC cut by the line A_1B_1 (fig. 5). We get the distance BK from the point of intersection K of the lines AB and A_1B_1 as $BK = \frac{AB \cdot A_1B}{|AC-BC|}$. We use proportion for the trapezium $ABB'A'$, we get the distance BK_1 from the point K_1, intersection of the lines AB and $A'B'$ as $BK_1 = \frac{AB \cdot A_1B}{|AC-BC|}$. Hence $BK = BK_1$ and the points K and K_1 are coincident. ■

Theorem 6 *The plane of the base triangle and the plane of the upper triangle intersect along the Gergonne line of the base triangle.*

Proof.

The line AB is a projection of the line $A'B'$ on the base plane. Hence the point of intersection of the lines AB and $A'B'$ lies on the line of intersection of the planes ABC and $A'B'C'$ (fig. 6). Similarly, K', point of intersection of the lines AC and $A'C'$, and K'', point of intersection of the lines BC and $B'C'$ lie on the line of intersection of the planes ABC and $A'B'C'$. So points K, K' and K'' are collinear. These points are also the crossing points of the lines AB, A_1B_1, and similar. According to the definition, these points lie on the polar of the Gergonne point of ABC, the point of intersection of the Cevians AA_1, BB_1 and CC_1.

Hence we proved that these points are collinear and lie on the Gergonne line of ABC. ■

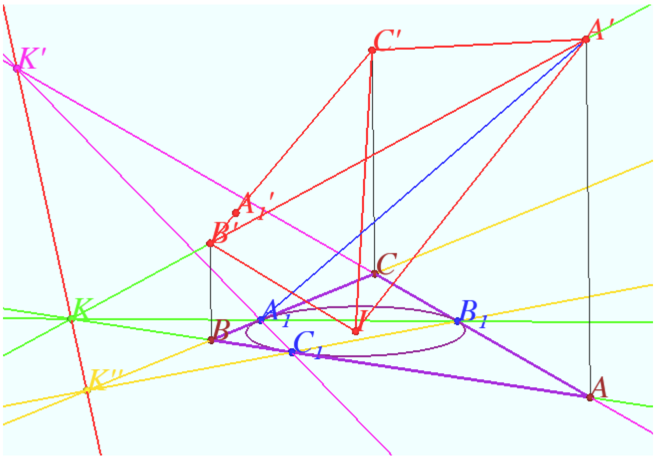


Figure 6: Intersection of the base plane and the upper plane

(Use mouse click to activate an interactive figure).

6 Soddy line an circles.

Theorem 7 *The Soddy line is perpendicular to the Gergonne line.*

This property is found in [6, p. 324 §6].

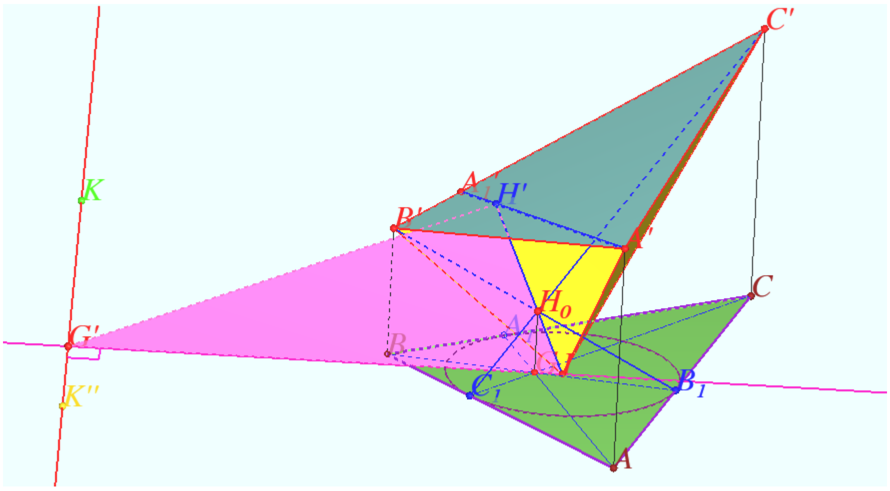


Figure 7: The Soddy line of a triangle is perpendicular to its Gergonne line

(Use mouse click to activate an interactive figure).

Proof.

By definition, the Soddy line contains the Gergonne point G and the incenter I [6, p. 319]. The plane IGH_0 is perpendicular to the base plane ABC because $GH_o \perp ABC$ (fig. 7). The plane IGH_0 is also perpendicular to the upper plane $A'B'C'$ because IH_o is an altitude of the orthocentric Pavillet tetrahedron. Hence the Gergonne line KK' which, from Theorem 6, belongs to both planes is perpendicular to the plane IGH_o and therefore to the line IG. ■

Theorem 8 *The Soddy inner and outer centers lie on the Soddy line.*

Proof.

Let S be the inner Soddy center (as defined in [2, § 2-3] or [5]). Let S_o belongs to the same half-space define by the base plane as the Pavillet tetrahedron, such that $SS_o \perp ABC$ and $SS_o = \rho$ where ρ is the radius of the inner Soddy circle. We call S_a the point of contact of the inner Soddy circle with the circle centered at A with radius x. The points A, S_a, S are collinear. The triangle S_oS_aA' is right, hence S_a

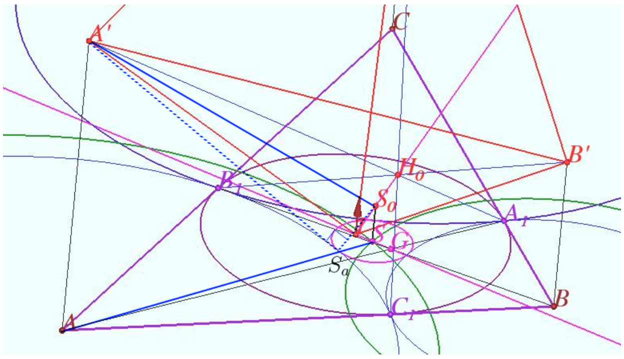
$$S_oA'^2 = S_oS_a^2 + S_aA'^2 = 2\,\rho^2 + 2\,x^2$$

and we find similarly S_oB' and S_oC'. Using (1), we find that

$$A'I^2 - B'I^2 = 2\,x^2 - 2\,y^2 = A'S_o^2 - B'S_o^2$$

and

$$A'I^2 - C'I^2 = 2\,x^2 - 2\,z^2 = A'S_o^2 - C'S_o^2.$$



So we proved that points I and S_o lie on the same perpendicular to the plane $A'B'C'$ (fig. 8). The point S being the projection of S_o on the plane ABC lies on the Soddy line, projection of IH_o on the base plane. The reader can make a similar proof for the outer Soddy center. ■

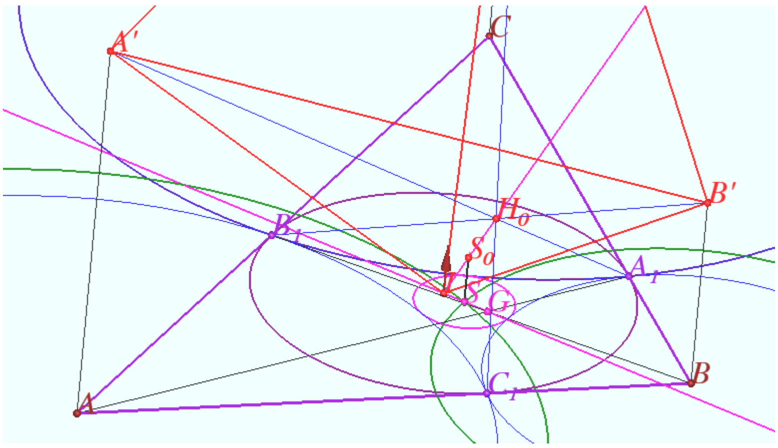


Figure 8: The Soddy inner center lies on the Soddy line

(Use mouse click to activate an interactive figure).

7 The circumcircle sphere

Definition 9 *We define the circumcircle sphere as the sphere having the circumcircle of the base triangle as diametral circle.*

Theorem 10 *The trace of the circumcircle sphere of the base triangle on the plane of the upper triangle is the Euler circle of this triangle.*

Proof.

Let C_2 be the midpoint of the side AB of the base triangle and C_2' be the midpoint of the side $A'B'$ of the upper triangle (fig. 9). Let O be the circumcenter of the base triangle so that $OC_2 \perp AB$. From the trapezium $AA''B'B$, we understand that

$$C_2C_2' = C_2A = \frac{x+y}{2}.$$

Hence, $OA = OC_2' = R$ where R is the radius of the circumcircle of the base triangle ABC. Similarly $OA_2' = OB_2' = R$. Therefore all midpoints of the sides of the upper triangle lie on the circumcircle sphere of the base triangle (fig. 10). ■

Corollary 11 *The centroid of the Pavillet tetrahedron lies on the line joining the circumcenter of the base triangle to the center of the Euler circle of the upper triangle.*

Proof.

The center of the first twelve point sphere of an orthocentric tetrahedron is its centroid [1, § 797-798]. Now, because the trace of the first twelve point sphere on the face of the upper triangle is also the Euler circle of this triangle, its center lies on a perpendicular to the upper face going through the center of this circle, but the center of the circumcircle sphere also lies on this perpendicular. ■

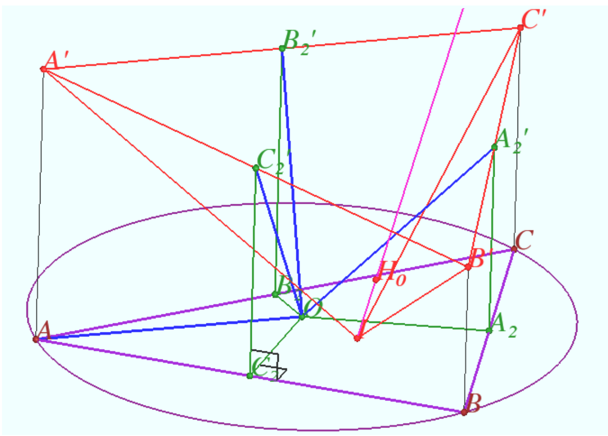


Figure 9: Radius of the circumcircle sphere

(Use mouse click to activate an interactive figure).

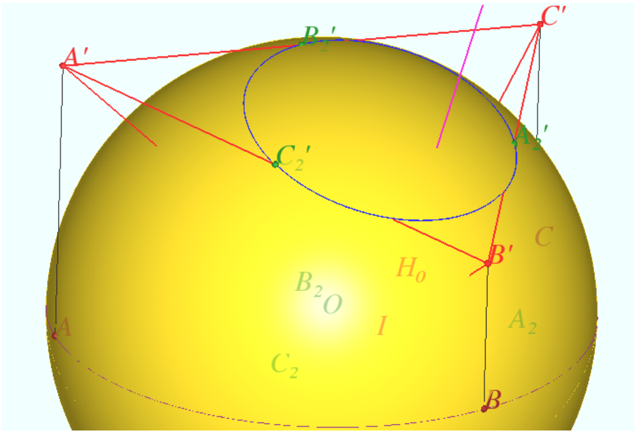


Figure 10: The Euler circle of the upper triangle is the trace of the circumcircle sphere of the base triangle on the upper plane
(Use mouse click to activate an interactive figure).

8 Conclusions

We have given elementary geometric or algebraic proofs of the fundamental properties of the Pavillet tetrahedron. Moreover we have found elementary proofs for a number of properties of the Soddy line of a triangle. Lots of geometric properties remained to be discovered about this new object and from the use of inversion or other transformations we probably could also derive interesting results.

Though it is not new to use solid geometry to solve plane geometry problems, it is rather unusual in triangle geometry (may be with the exception of [4, § 486 p. 290]). Hence the discovery of the Pavillet tetrahedron gives a new dimension to triangle geometry. There is not only a correspondence between the base triangle and the tetrahedron, we have also found a correspondence between the base triangle and the upper triangle. The Pavillet tetrahedron could be a new type of triangle transformation which would need to be formalized. In the meantime, it allows finding new short ways to prove some well known results about triangle properties usually found with quite labor-intensive methods. Students which study geometry, get one more tool to address known problems in an unusual way and to demonstrate the unity of the world of mathematics.

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Software Packages
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